USE OF A VARIATIONAL METHOD TO SOLVE A HEAT CONDUCTION PROBLEM WITH INTERNAL HEAT SOURCES

A. M. Fain

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An examination is made of Kantorovich's variational method for analytical solution of steady heat conduction problems with internal heat sources having a two-dimensional distribution law. A method is described for choosing coordinate functions which satisfy the assigned boundary condition. Dimensionless coefficients of the system of Euler equations are introduced.

In order to formulate a whole series of topical optimal problems it is necessary to have available a simple, but sufficiently accurate analytical expression to describe the temperature field. The presence of internal heat sources with a complex two-dimensional distribution and nontrivial boundary conditions makes analytical solution of the heat conduction equation by ordinary methods difficult, and the result obtained unwieldy. The approximate variational method examined in this paper allows solution of such problems with minimum expenditure of effort.

The heat conduction equation for a cylinder (Fig. 1) with internal heat sources under steady conditions and axial symmetry has the form

$$\frac{\partial^2 T}{\partial x^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}+\frac{q_v}{\lambda}=0. \quad (1)$$

Let the internal heat sources be assigned in the form

$$q_v=q_0 w_1(x)\, w_2(r), \quad (2)$$

where q_0 is the internal specific heat generation at the point with coordinates $x = 0$, $r = 0$; $w_1(x)$, $w_2(r)$ are certain given functions.

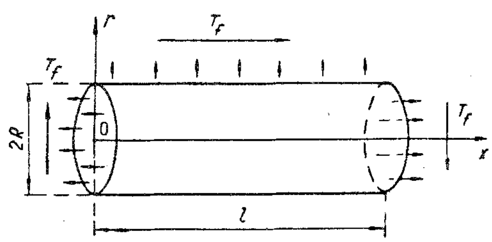

Fig. 1. Schematic of solid cylinder.

The boundary conditions are assigned in the following form:

$$x=0, \quad \lambda\frac{\partial T}{\partial x}=\alpha_1(T-T_f),$$

$$x=l, \quad -\lambda\frac{\partial T}{\partial x}=\alpha_2(T-T_f),$$

$$r=R, \quad -\lambda\frac{\partial T}{\partial r}=\alpha_3(T-T_f). \quad (3)$$

We introduce the dimensionless coordinates

$$r/R=\rho, \quad x/R=\psi, \quad (T-T_f)/T_f=t. \quad (4)$$

We also denote

$$\alpha_1 R/\lambda=k_1, \quad \alpha_2 R/\lambda=k_2, \quad \alpha_3 R/\lambda=k_3, \quad R^2 q_0/\lambda T_f=q. \quad (5)$$

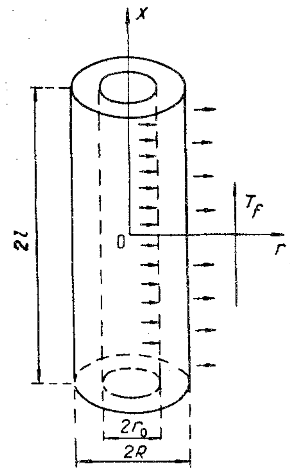

Fig. 2. Schematic of hollow cylinder.

Then (1) takes the form

$$\frac{\partial^2 t}{\partial \psi^2}+\frac{1}{\rho}\frac{\partial t}{\partial \rho}+\frac{\partial^2 t}{\partial \rho^2}+q w_1 w_2=0. \quad (6)$$

The boundary conditions may correspondingly be written as

$$\psi=0, \quad \frac{\partial t}{\partial \psi}-k_1 t=0; \quad (7)$$

$$\psi=\frac{l}{R}, \quad \frac{\partial t}{\partial \psi}+k_2 t=0; \quad (8)$$

$$\rho=1, \quad \frac{\partial t}{\partial \rho}+k_3 t=0. \quad (9)$$

Equation (6) is the Euler-Ostrogradskii equation

$$F'_t-\frac{\partial}{\partial \psi}F'_{t'_\psi}-\frac{\partial}{\partial \rho}F'_{t'_\rho}=0 \quad (10)$$

for the functional

$$v[t(\psi,\rho)]=\int\limits_{\rho=0}^{\rho=1}\int\limits_{\psi=0}^{\psi=l/R}\left[\left(\frac{\partial t}{\partial \psi}\right)^2+\left(\frac{\partial t}{\partial \rho}\right)^2-2tqw_1w_2\right]\rho\, d\rho\, d\psi, \quad (11)$$

where F is the integrand function.

The solution of (6) may approximately be represented by the sum

$$t=\sum_{i=1}^{n}u_i(\rho)\cdot s_i(\psi), \quad (12)$$

where $s_i(\psi)$ are functions subject to definition from the condition that functional (11) attains an extremum; $u_i(\rho)$ are coordinate functions which must be chosen such that, firstly, boundary condition (9) and the symmetry conditions

$$\rho = 0, \quad \frac{\partial t}{\partial \rho} = 0 \tag{13}$$

are satisfied, and, secondly, the nature of the variation of temperature along a radius is described in a manner close enough to the actual.

The more successful is the choice of coordinate functions u_i, the fewer terms are required in (12) to obtain a satisfactory approximation. We shall determine function u_i as a solution of the corresponding one-dimensional equation

$$\frac{d^2u_1}{d\rho^2} + \frac{1}{\rho}\frac{du_1}{d\rho} + w_2(\rho) = 0. \tag{14}$$

The first integration gives

$$\frac{du_1}{d\rho} = -\frac{1}{\rho}\int w_2(\rho)\rho\, d\rho + \frac{C_1}{\rho}. \tag{15}$$

It follows from condition (9) that $C_1 = 0$. We introduce the auxiliary function Φ:

$$\frac{d\Phi}{d\rho} = \frac{1}{\rho}\int w_2(\rho)\rho\, d\rho. \tag{16}$$

As a result of a second integration, using boundary condition (9), we find

$$u_1 = \Phi(1) - \Phi(\rho) + \frac{1}{k_3}\frac{d\Phi(1)}{d\rho}. \tag{17}$$

It is not difficult to see that if u_i is built up in the form

$$u_i = \left[\Phi(1) - \Phi(\rho) + \frac{i}{k_3}\frac{d\Phi(1)}{d\rho}\right]^i, \tag{18}$$

then conditions (9) and (13) will be satisfied for any i.

In accordance with the basic idea of the variational method applied, expression (12) should be substituted in the functional (11) subject to minimization. Subsequently, carrying out integration with respect to ρ, we choose the unknown functions s_i from the condition of minimization of the functional obtained, which now contains a function of only one variable ψ; this leads to solution of a system of ordinary differential equations. We have

$$v = \int_0^1\int_0^{l/R}\left[\left(\sum_{i=1}^{n}\frac{\partial s_i}{\partial\psi}u_i\right)^2 + \left(\sum_{i=1}^{n}s_i\frac{\partial u_i}{\partial\rho}\right)^2 - 2qw_1w_2\sum_{i=1}^{n}s_iu_i\right)\rho\, d\rho\, d\psi. \tag{19}$$

We introduce the designation

$$a_i = \int_0^1 w_2(\rho)u_i\rho\, d\rho, \quad a_{ij} = \int_0^1 u_iu_j\rho\, d\rho,$$

$$b_{ij} = \int_0^1 \frac{\partial u_i}{\partial\rho}\frac{\partial u_j}{\partial\rho}\rho\, d\rho, \tag{20}$$

where i, j = 1, 2, ..., n.

Integrating with respect to ρ in the functional (19) within the limits indicated, we obtain

$$v = \int_0^{l/R}\left[\sum a_{ii}\left(\frac{\partial s_i}{\partial\psi}\right)^2 + 2\sum a_{ij}\frac{\partial s_i}{\partial\psi}\frac{\partial s_j}{\partial\psi} + \sum b_{ii}s_i^2 + 2\sum b_{ij}s_is_j - 2qw_1\sum a_is_i\right]d\psi, \tag{21}$$

where the summation is performed for i, j = 1, 2, ..., n, but in the range of one sum, $i \neq j$. Functional (21) must realize an extremum on the 2n-parametric family of curves determined from the system of Euler equations

$$F'_{s_i} - \frac{d}{d\psi}F'_{s'_i} = 0, \tag{22}$$

which may be written in the general case as

$$\sum_{j=1}^{n}a_{ij}s''_j - \sum_{j=1}^{n}b_{ij}s_j = -qw_1a_i, \tag{23}$$

where i = 1, 2, ..., n.

Example. To find the steady temperature distribution in a solid cylinder in which the heat source generation law is given by the relation

$$q_v = q_0\exp(-\mu_x x)\left(1 - \frac{q_R}{q_0}\frac{r^2}{R^2}\right).$$

The boundary conditions are assigned in the form (3). We designate

$$q_R/q_0 = \beta, \quad \mu_x R = \mu;$$

then

$$w_1 = \exp(-\mu\psi), \quad w_2 = 1 - \beta\rho^2.$$

We determine the auxiliary function

$$\frac{d\Phi}{d\rho} = \frac{1}{\rho}\int w_2\rho\, d\rho = \frac{\rho}{2}(1 - \beta\rho^2),$$

$$\Phi = \frac{\rho^2}{4}\left(1 - \frac{\beta}{4}\rho^2\right).$$

Let us restrict ourselves to one term of series (12); then a solution of the original equation (6) should be sought in the form

$$t = \left[(1-\rho^2) - \frac{\beta}{4}(1-\rho^4) + \frac{2-\beta}{k_3}\right]s_1.$$

The system (23) is one equation, in which the coefficients with mixed indices are equal to zero. We find

$$a_{11} = \int_0^1 u_1^2\rho\, d\rho = \frac{1}{2}\frac{k_3^2 + 6k_3 + 12}{3k_3^2};$$

$$b_{11} = \int_0^1\left(\frac{\partial u_1}{\partial\rho}\right)^2\rho\, d\rho = \frac{1}{2}\frac{\beta^2}{4};$$

$$a_1 = \int_0^1 w_2 u_1 \rho \, d\rho = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{3}\beta + \right.$$

$$\left. + \frac{1}{16}\beta^2 + \frac{2-\beta}{k_3}\left(1 - \frac{\beta}{2}\right)\right].$$

The Euler equation in this case takes the form

$$a_{11}s_1'' - b_{11}s_1 = -q\exp(-\mu\psi)a_1.$$

Its solution is

$$s_1 = \frac{a_1 q}{\mu^2 a_{11} - b_{11}}[C_1\exp(\omega\psi) + C_2\exp(-\omega\psi) + \exp(-\mu\psi)],$$

where $\omega = \sqrt{b_{11}/a_{11}}$.

Thus, the desired temperature distribution is described by the expression

$$t = \frac{a_1 q}{\mu^2 a_{11} - b_{11}}\left[(1-\rho^2) - \frac{\beta}{4}(1-\rho^4) + \frac{2-\beta}{k_3}\right] \times$$

$$\times [C_1\exp(\omega\psi) + C_2\exp(-\omega\psi) + \exp(-\mu\psi)].$$

The constants of integration are determined from boundary conditions (7) and (8).

We shall examine the problem of heat conduction for a hollow cylinder (Fig. 2). Let the boundary conditions be assigned in the following form:

$$x = l, \ \lambda\frac{\partial T}{\partial x} = 0,$$

$$x = -l, \quad -\lambda\frac{\partial T}{\partial x} = 0,$$

$$r = r_0, \quad -\lambda\frac{\partial T}{\partial r} = Q,$$

$$r = R, \quad -\lambda\frac{\partial T}{\partial r} = \alpha_3(T - T_f). \tag{24}$$

To the designations of (5) we add

$$r_0/R = \rho_0, \quad QR/T_f\lambda = W. \tag{25}$$

Then the boundary conditions, allowing for symmetry, finally take the following form:

$$\psi = \frac{l}{R}, \quad \frac{\partial t}{\partial\psi} = 0; \tag{26}$$

$$\psi = 0, \ \frac{\partial t}{\partial\psi} = 0; \tag{27}$$

$$\rho = \rho_0, \ \frac{\partial t}{\partial\rho} = -W; \tag{28}$$

$$\rho = 1, \ \frac{\partial t}{\partial\rho} + k_3 t = 0. \tag{29}$$

We seek a solution in the form

$$t = \sum_{i=1}^{n} u_i(\rho)s_i(\psi) + \Delta. \tag{30}$$

The introduction of Δ permits us to replace the inhomogeneous boundary condition (28) by a homogeneous one,

$$\rho = \rho_0, \ \frac{\partial}{\partial\rho}\sum_{i=1}^{n} u_i s_i = 0. \tag{31}$$

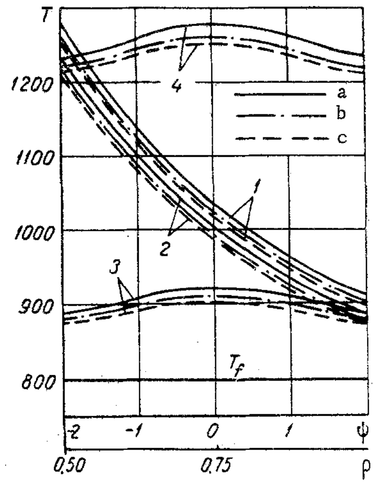

Fig. 3. Distribution of temperature, T (°K), in the hollow cylinder: 1 and 2) radial variation of temperature with $\psi = 0$ and $\psi = l/R = 2$; 3 and 4) axial distribution with $\rho = \rho_0 = 0.5$ and $\rho = 1$; a) first approximation; b) second; c) third.

We define Δ in the binomial form $\Delta = A\rho + B\rho^2$, in which we shall find A and B, using the obvious conditions

$$\rho = \rho_0, \ \frac{d\Delta}{d\rho} = -W;$$

$$\rho = 1, \ \frac{d\Delta}{d\rho} + k_3\Delta = 0.$$

Then

$$\Delta = W\frac{\rho^2(1+k_3) - \rho(2+k_3)}{2(1-\rho_0) + k_3(1-2\rho_0)}. \tag{32}$$

Retaining all the arguments presented above, we determine u_1 as the solution of (14) with boundary conditions (28) and (29):

$$u_1 = \Phi(1) - \Phi(\rho) + \rho_0\frac{d\Phi(\rho_0)}{d\rho}\ln\rho +$$

$$+ \frac{1}{k_3}\left[\frac{d\Phi(1)}{d\rho} - \rho_0\frac{d\Phi(\rho_0)}{d\rho}\right]. \tag{33}$$

It is then necessary to seek the solution of (6) for the hollow cylinder in the form

$$t = \sum_{i=1}^{n}\left\{\Phi(1) - \Phi(\rho) + \rho_0\frac{d\Phi(\rho_0)}{d\rho}\ln\rho + \frac{i}{k_3}\times\right.$$

$$\left.\times\left[\frac{d\Phi(1)}{d\rho} - \rho_0\frac{d\Phi(\rho_0)}{d\rho}\right]\right\}_i s_i + \Delta. \tag{34}$$

Substituting (30) into functional (11), we have

$$v = \int_0^{l/R}\int_{\rho_0}^{1}\left[\left(\sum_{i=1}^{n}\frac{\partial s_i}{\partial\psi}u_i\right)^2 + \left(\sum_{i=1}^{n}s_i\frac{\partial u_i}{\partial\rho} + \frac{d\Delta}{d\rho}\right)^2 - \right.$$

$$\left. - 2qw_1w_2\left(\sum_{i=1}^{n}s_iu_i + \Delta\right)\right]\rho\,d\rho\,d\psi. \tag{35}$$

We add one more designation to (20):

$$b_i = \int_{\rho_0}^{1} \frac{d\Delta}{d\rho} \frac{\partial u_i}{\partial \rho} \rho\, d\rho. \quad (36)$$

In functional (35) we carry out integration with respect to ρ in the limits indicated. The Euler equations then take the form

$$\sum_{j=1}^{n} a_{ij} s_j'' - \sum_{j=1}^{n} b_{ij} s_j = b_i - q w_1 a_i \ (i = 1, 2, \ldots, n). \quad (37)$$

Example. To find the steady temperature distribution in a hollow cylinder with boundary conditions (24) and the following initial data: R = 0.10 m; λ = 50 kcal/m • hr• °C; Q = $3 \cdot 10^6$ kcal/m^2• hr; r_0 = 0.5 m; l = 0.20 m; α_3 = 2500 kcal/m^2 • hr• °C; w_1 = cos 0.3 x/R; T_f = 800° K; $q_0 = 5 \cdot 10^6$ kcal/m^3 • hr; w_2 = 1 − (3/8)(r/R).

In dimensionless coordinates the starting data take the following form:

$$W = 0.75;\ q = 1.25,\ w_1 = \cos 0.3\psi,\ w_2 = 1 - \frac{3}{8}\rho,\ k_3 = 5,$$
$$\Delta = -0.75(7\rho - 6\rho^2).$$

We find the auxiliary function (the constant multiplier 1/4 has been dropped)

$$\frac{d\Phi}{d\rho} = 2\rho - \frac{1}{2}\rho^2, \quad \Phi = \rho^2 < \frac{1}{6}\rho^3.$$

Then

$$\frac{d\Phi(1)}{d\rho} = \frac{3}{2}, \quad \frac{d\Phi(0.5)}{d\rho} = 0.8750, \quad \Phi(1) = 0.8333.$$

The general expression for function u_i, in agreement with (34), will have the form

$$u_i = \left(0.8333 - \rho^2 + \frac{1}{6}\rho^3 + 0.4375 \ln \rho + i \cdot 0.2125\right)^i.$$

The coefficients of the system of Euler equations are

$a_1 = 0.10595$	$a_{11} = 0.05695$	$b_1 = -0.65196$	$b_{11} = 0.19314$
$a_2 = 0.10346$	$a_{12} = 0.05739$	$b_2 = -0.72831$	$b_{12} = 0.21623$
$a_3 = 0.15753$	$a_{22} = 0.05614$	$b_3 = -1.06082$	$b_{22} = 0.22338$
	$a_{13} = 0.09018$		$b_{13} = 0.25843$
	$a_{23} = 0.08556$		$b_{23} = 0.29621$
	$a_{33} = 0.13018$		$b_{33} = 0.37951.$

With one term of the series we obtain

$$a_{11} s_1'' - b_{11} s_1 = b_1 - a_1 \cdot 1.25 \cos 0.3\psi.$$

Its general solution is

$$s_1 = C_1 \exp(\omega\psi) + C_2 \exp(-\omega\psi) - \frac{b_1}{b_{11}} +$$
$$+ \frac{1.25 a_1}{0.3^2 a_{11} + b_{11}} \cos 0.3\psi,$$

where $\omega = \sqrt{b_{11}/a_{11}}$.

The temperature distribution law takes the form

$$t_1 = \left(0.8333 - \rho^2 + \frac{1}{6}\rho^3 + 0.4375 \ln \rho +\right.$$
$$\left. + 0.2125\right)\{1.57 w^{-3} [\exp(1.84\psi) +$$
$$+ \exp(-1.84\psi)] + 3.37 + 0.67 \cos 0.3\psi\} - 0.75(7\rho - 6\rho^2).$$

With two terms of the series we obtain the system

$$a_{11} s_1'' + a_{12} s_2'' - b_{11} s_1 - b_{12} s_2 = b_1 - 1.25 a_1 \cos 0.3\psi,$$
$$a_{21} s_1'' + a_{22} s_2'' - b_{21} s_1 - b_{22} s_2 = b_2 - 1.25 a_2 \cos 0.3\psi.$$

The temperature distribution law is

$$t_2 = \left(0.8333 + \rho^2 + \frac{1}{6}\rho^3 + 0.4375 \ln \rho + 1 \cdot 0.2125\right) \times$$
$$\times \{4.33 \cdot 10^{-3} [\exp(1.76\psi) + \exp(-1.76\psi)] -$$
$$- 0.301 \cos 4.85\psi + 0.342 \cdot 10^{-2} \sin 4.85\psi +$$
$$+ 2.72 - 0.201 \cos 0.3\psi\} + \left(0.8333 - \rho^2 + \frac{1}{6}\rho^3 +\right.$$
$$\left. + 0.4375 \ln \rho + 2 \cdot 0.2125\right)^2 \times \{1.74 \cdot 10^{-2} [\exp(1.76\psi) +$$
$$+ \exp(-1.76\psi)] - 0.252 \cos 4.85\psi +$$
$$+ 0.285 \cdot 10^{-2} \sin 4.85\psi + 2.10 +$$
$$+ 0.466 \cos 0.3\psi\} - 0.75(7\rho - 6\rho^2).$$

The curves of Fig. 3 were drawn from the calculated data. The maximum discrepancy of the results of the first and second approximations was 15° (15/ /800 • 100 = 1.9%), and the discrepancy of the results of the second and third approximations was 5–6° (6/ /800 • 100 = 0.75%).

The variational method examined, which reduces the solution of the two-dimensional heat conduction equation to the solution of a system of linear differential equations, is very simple and rapidly convergent. The proposed means of choosing the coordinate functions, and the possibility of using even simpler integrators to calculate the coefficients of the system of Euler equations, make this method technically suitable.

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